H-2-30-22 Roll No. ....

II Semester Examination, 2022

M.Sc.

MATHEMATICS

Paper I

(Advanced Abstract Algebra-II)

Time : 3 Hours] [Max. Marks : 80

Note : *All Questions are Compulsory. Question Paper comprises of 3 sections.* ***Section 'A'*** *is Objective type/Multiple Choice questions with no internal Choice.* ***Section 'B'*** *is Short answer type with internal Choice.* ***Section 'C'*** *is Long answer type with internal Choice.*

Section 'A' **1 × 10 = 10**

(Objective Type Questions)

Choose the correct answer :

1. The condition when the left R module becomes the right R module is :

(a) R is a commutative ring

(b) R is a ring with unity

(c) Every element of R has a multiplicative inverse

(d) None of the above

P.T.O.

2. An R module M which is a direct sum of simple submodule is called :

(a) Simple module

(b) Semi simple module

(c) Free module

(d) Minimal sub module

3. Let $U = \{v \in V : vT = 0\}$ be a subspace of V if $s(T)$ is the range of T and $\eta(T)$ be dimension of U then following will be true :

(a) $r(T) = \eta(T) = 0$

(b) $r(T) + \eta(T) = \dim V$

(c) $r(T) - \eta(T) = 0$

(d) $r(T)\ \eta(T) = 0$

4. Two Nilpotent linear transformations are similar if and only if they have :

(a) Different Invariant

(b) Same Invariant

(c) Non-equivalence relation

(d) Same antisymmetric relation

5. The index of the Nilpotent matrix

$$A = \begin{bmatrix} -2 & 1 & 1 \\ -3 & 1 & 2 \\ -2 & 1 & 1 \end{bmatrix} \text{ is :}$$

(a) 1

(b) 2

(c) 3

(d) None of the above

6. If $f(x)$ is the characteristic polynomial of a linear operator T, then $f(T) = 0$ is known as :

(a) Hilbert basis theorem

(b) Noether-Lasker theorem

(c) Cayley-Hamilton theorem for linear operator

(d) None of the above

7. The index of Nilpotency of $A = \begin{pmatrix} 0, 1 \\ 0, 0 \end{pmatrix}$ is :

(a) 2 (b) 1

(c) 4 (d) 3

8. If $\delta : N \times N \to (0, 1)$ defined as

$$\delta_{jk} = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \neq k \end{cases}$$

then value of e_{ij}^2 is :

(a) 0 (b) 1

(c) e_{ii} (d) e_{ij}

9. Companion matrix of $(x + 1)^3$ is :

(a) $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}$

10. Let M be a non-zero module and any two non-zero submodules of M have non-zero intersection then M is called :

(a) Uniform module

(b) Primary module

(c) P-primary module

(d) None is correct

Section 'B' **4 × 5 = 20**

(Short Answer Type Questions)

Note : *Attempt one question from each unit.*

Unit–I

1. If M be a simple R module then prove that $\text{Hom}_R (M, M)$ is a division ring.

Or

Let R be a commutative ring with unity and $e \neq 0, 1$ be an idempotent, then prove that Re cannot be a free R-module.

Unit–II

2. Let A be an algebra, with unit element over F and let dimension of A over f be m. Then prove every element in A satisfies some non-trivial polynomial in $f(x)$ of degree at most m.

Or

Let V be an n-demensional vector space over F and $T \in A(V)$ has all its characteristic roots in F. Then prove T satisfies a polynomial of degree n over F.

Unit–III

3. Define similarity of a linear transformation and prove that similarity of linear transformation is an equivalence relation.

Or

Find the Jordan Canonical form of

$$A = \begin{bmatrix} 5 & 1 & -2 & 4 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Unit–IV

4. Obtain the smith normal form and rank of the following matrix

$$A = \begin{bmatrix} 0 & 2 & -1 \\ -3 & 8 & 3 \\ 2 & -4 & -1 \end{bmatrix}$$

Or

Let M be a finitely generated module over a principal ideal domain R then prove that

$$M = F \oplus T \text{ or } M$$

Where (i) $F \cong R^S$ for same non-negative integer S and

(ii) T or $M \cong \frac{R}{Ra_1} \oplus \frac{R}{Ra_2} \oplus ..... \oplus \frac{R}{R_{ar}}$

Where ai are non-zero non unit elements in R such that $a_1 \mid a_2 \mid a_3 .... \mid a_r$.

Unit–V

5. Write the rational canonical form of 6×6 matrix whose invariant factors are

$(x + 3), (x + 3)(x + 1), (x + 3)(x + 1)^2$

Or

Find the rational canonical form of

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Section 'C' **10 × 5 = 50**

(Long Answer Type Questions)

Note : *Attempt one question from each unit.*

Unit–I

1. Let R be a ring with unity and M be an R module. Then prove that the following are equivalent :

(i) M is simple

(ii) $M \neq (0)$, and M is generated by any $0 \neq x \in M$

(iii) $M \cong \frac{R}{I}$ where I is a maximal left ideal of R.

Or

If J is a nil left ideal in an artinian ring R, the prove that J is Nilpotent.

Unit–II

2. If $u \in V_1$ is such that $uT^{n_1-k} = 0$ where $0 < k \leq n$, then prove that $u = u_0T^k$ for some $u_0 \in V_1$.

Or

Let A be an algebra with unit element over F, then prove that A is isomorphic to a subalgebra of $A(V)$ for some vector space V over F.

Unit–III

3. Prove that two Nilpotent transformations $S, T \in A(V)$ are similar if and only if they have the same invariants.

Or

Define the Jordan Canonical form of a square matrix and prove that A Jordan block J may be written as the sum of a scalar matrix and a Nilpotent.

Unit–IV

4. Find the smith normal form and rank for the following matrix over PID R.

$$\begin{bmatrix} -x-3 & 2 & 0 \\ 1 & -x & 1 \\ 1 & -3 & -x-2 \end{bmatrix} \text{ where } R = \phi(x)$$

Or

Let R be a principal ideal domain and M be any finitely generated R module then prove

$$M \cong R^S \oplus \frac{R}{Ra_1} \oplus \frac{R}{Ra_2} \oplus \ldots\ldots \oplus \frac{R}{R_{ar}}$$

a direct sum of cyclic modules, where the a_i are non-zero, non units and

$$a_i \mid a_{i+1}, i = 1, 2\ldots\ldots, r-1$$

Unit–V

5. Find the abelian group generated by (x_1, x_2, x_3) subject to

$$5x_1 + 9x_2 + 5x_3 = 0$$

$$2x_1 + 4x_2 + 2x_3 = 0$$

$$x_1 + x_2 - 3x_3 = 0$$

Or

Reduce the following matrix to rational canonical form

$$A = \begin{bmatrix} -3 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & -3 & -2 \end{bmatrix}$$

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