H-4/25/22

Roll No. ....

IV Semester Examination, 2022

M.Sc.

MATHEMATICS

Paper III

(WAVELETS-II)

Time : 3 Hours] [Max. Marks : 80

Note : *All questions are compulsory. Question Paper comprises of 3 sections. Section **A** is objective type/multiple choice questions with no internal choice. Section **B** is short answer type with internal choice. Section **C** is long answer type with internal choice.*

SECTION A **1×10=10**

(Very Short Answer Type Questions)

1. Define low pass filter.
2. Define frame operator.
3. Write the statement of Balian low the orem for frames.
4. Define Dual light frame for $L^2(R)$.
5. What do you mean by Fast Fourier transform ? (write in 2-3 lines only).

P.T.O.

6. Define characterization of MRA wavelets.
7. Write the condition that "a frame is light".
8. What do you mean by a projection of $f \in L^2(R)$ onto V_j.
9. Write an expression of window function W_j to express discrete version of cosine transform.
10. Define translation and dilations of a single function $\psi \in L^2(R)$.

SECTION B **4×5=20**

(Short Answer Type Questions)

Note : *Attempt one question from each unit.*

Unit-I

1. Suppose that $\{e_j : j = 1, 2, 3 \ldots\}$ is a system of vectors in a Hilbert space H satisfying condition

   $\|f\|^2 = \sum_{j=1}^{\infty} |< f, e_j >|^2$ holds for all $f \in H$.

   If $\| e_j \| \geq 1$ for $j = 1, 2, 3\ldots$.

   Show that $\{e_j : j = 1, 2, 3\ldots\}$ is an orthogonal basis for H.

   Or

   If ψ is an orthonormal wavelet and $|\hat{\psi}|$ is continuous at zero, then show that $\hat{\psi}(0) = 0$.

Unit-II

2. Let P_j be the projection onto V_j, the show that $\| P_j f - f \|_2^2 = \| f \|^2 - \| P_j f \|_2^2 \to 0$ as $j \to \infty$.

Or

Let m_0 be a 2π-periodic function in the class C^n, $n \in N$, which satisfies the condition $m_0(0) = 1$ and $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1$ for $a.e.$, $\xi \in R$. Let $\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi)$, then show that $\hat{\varphi}$ and all its derivative upto order n, in the sense of distributions belong to $L^2(R)$.

Unit-III

3. If $\{\phi_j : j \in J\}$ is a frame on a Hilbert space H with frame bounds A and B. Then the collection

$$\{\tilde{\phi}_j \equiv S^{-1}(\phi_j) : j \in J\}$$

is also a frame for H with frame bound $\frac{1}{B}$ and $\frac{1}{A}$, prove it.

Or

Show that if $0 < \epsilon \le \frac{\pi}{3}$, then the system $\{(\psi^{\epsilon})_{j,k} : j, k \in Z\}$ is a frame for $H^2(R)$.

Unit-IV

4. If the function E_k where $E_k(r) = r_k$ are the analogues of the exponentials $e^{2\pi i k\theta}$ in the case of classical Fourier series. Then show that $\left\{\frac{1}{\sqrt{N}} E_k : k = 0, 1 \ldots. N-1\right\}$ is an orthonormal basis for $L^2(\sqrt{N})$.

Or

Show that $\tilde{y}_k$ equals $\frac{1}{2\pi i} e^{\pi i \frac{k}{2N}}$ times the *DCT* coefficients $\alpha_k(N)$ for the function f.

Unit-V

5. Write in details "How the Haar wavelet works for doing the decomposition algorithm."

Or

Explain reconstruction algorithm for wavelets in one dimensional case.

SECTION C 10×5=50

(Long Answer Type Questions)

Note : *Attempt one question from each unit with internal choice.*

Unit-I

1. Show that

$$\sum_{j \in Z} 2^{-j} \int_R | \hat{f}(2^{-j}\xi) \hat{\psi}(\xi) |$$

$$\sum_{k \neq 0} | \hat{f}[2^{-j}(\xi + 2k\pi)] \hat{\psi}(\xi + 2k\pi) | d\xi < \infty.$$

Or

Let $\psi \in L^2(R)$ be such that $|\hat{\psi}| = \chi_k$ for some measurable $k \in R$. Then show that ψ is an *MSF* wavelet if and only if

$$\sum_{j \in Z} |\hat{\psi}(2^j \xi)|^2 = 1 \text{ for } a.e.\ \xi \in R$$

and $$\sum_{j=0}^{\infty} \hat{\psi}(2^j \xi) \hat{\psi}[2^j(\xi + 2m\pi)] = 0$$

for $a.e.\ \xi \in R$, $m \in 2Z + 1$

Unit-II

2. Suppose that the low pass filter m_0 of an MRA is a C^1 function and the scaling function ψ satisfies $\hat{\psi}(0) = 1$ and $|\hat{\psi}(\xi)| = O(|\xi|^{-\frac{1}{2}-\alpha})$ at ∞ for some $\alpha > 0$. Then show that m_0 must satisfy the following property.

(i) There exists a set $k < R$ which is a finite. Union of closed bounded intervals such that 0 is in the interior of k

$$\sum_{k \in Z} \chi_k(\xi + 2k\pi) = 1 \text{ for } a.e.\ \xi \in R.$$

(ii) $m_0(2^{-j}\xi) \neq 0$ for all $j = 1, 2, 3....$ and all $\xi \in k$.

Or

Show that a function $\psi \in L^2(R)$ is a scaling function for an *MRA* if and only if

$$\sum_{k \in Z} |\hat{\psi}(\xi + 2k\pi)|^2 = 1 \text{ for } a.e.\ \xi \in T$$

$$\lim_{j \to \infty} |\hat{\psi}(2^{-j}\xi)| = 1 \text{ for } a.e.\ \xi \in R.$$

These exists a 2π-periodic function m_0 such that

$$\hat{\psi}(2\xi) = m_0(\xi) \hat{\psi}(\xi).$$

Unit-III

3. For any $h \in L^2(R)$, if $Qh \in L^2(R)$ and $Ph \in L^2(R)$, then show that

(i) $R(Qh)_{(s,t)} = s(Rh)_{(s,t)} + \frac{1}{2\pi i} \frac{\partial}{\partial t}(Rh)_{(s,t)}$

(ii) $R(Ph)_{(s,t)} = -i \frac{\partial}{\partial s}(Rh)_{(s,t)}$

Or

Let $\psi \in L^2(R)$ be such that

$$A_\psi = \underline{S}_\psi - \sum_{9 \in 2z+1} [\beta_\psi(9) \beta_\psi(-9)]^{1/2} > 0$$

and $B_\psi = \bar{S}_\psi + \sum_{9 \in 2z+1} [\beta_\psi(9)\, \beta_\psi(-9)]^{1/2} < \infty$

then show that $\{\psi_{j,k} : j, k \in Z]$ is a frame with frame bands A_ψ and B_ψ.

Unit-IV

4. If $N = 2^9$ then prove that $C_N \equiv E_1, E_2, E_3 \ldots. E_9$ where each E_j is an $N \times N$ matrix such that each row has precisely two non-zero entries.

Or

Explain discrete cosine transform and how will you differentiate it from fast cosine transform ?

Unit-V

5. Show that $l^2(z)$ is an orthonormal direct sum of the sequence E_j.

Or

Prove that the sequence $\{u_{j,k} : j \in Z, 0 \le k \le l_j - 1\}$ is an orthonormal basis for $l^2(Z)$, where

$$u_{j,k}(x) = \sqrt{\frac{2}{l_j}}\, \omega_j(x) \cos\left(\pi\left(k + \frac{1}{2}\right)\left(\frac{x - a_j}{l_j}\right)\right) j;\ x \in Z$$

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