G-3/378/22 Roll No. ....

III Semester Examination, January 2022

M.Sc.

MATHEMATICS

Paper III

(Wavelets-I)

Time : 3 Hours] [Max. Marks : 80

Note : *All questions are compulsory. Question Paper comprises of 3 Sections. Section A is objective type/multiple choice questions with no internal choice. Section B is short answer type with internal choice. Section C is long answer type with internal choice.*

SECTION A 1×10=10

(Objective Choice Questions)

Note : *Answer all questions.*

1. Write reconstruction formula for generating wavelet by a single function.

2. Define Lebesrgue point.

3. Write a necessary and sufficient condition for $\{e^{2\pi \sin x} b \cos\}_{\ln\in Z}$ to be an orthonormal system in $L^2(R)$.

4. Define smooth projections on $L^2(R)$.

5. State Plancherel theorem.

6. Prove the following property of low pass filter :

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1 \text{ for } a.e. \ \xi \in R$$

7. Write necessary and sufficient conditions for the orthonormality of the system $\{\psi_{j,k} : j, k \in Z\}$.

8. Write the conditions that completely characterize orthonormal wavelets.

9. Prove that the basic spline of order n, Δ^n, satisfies the following property : supp. $(\Delta^n) = [0, n + 1]$ and $\Delta^n(x) > 0$ for all $x \in (0, n + 1)$.

10. Write the Franklin periodic wavelet basis.

SECTION B 5×4=20

(Short Answer Type Questions)

Note : *Answer the following questions.*

Unit-I

1. Prove that, for $g = \chi_{[0, 1]}$, $\{g_{m, n} : m, n \in Z\}$ is an orthonormal basis of $L^2(R)$.

Or

Draw graph of the bell function b associated with $[\alpha, \beta]$ and prove that supp. $(b) \leq [\alpha - \epsilon, \beta + \epsilon']$ on $[\alpha - \epsilon, \alpha + \epsilon]$.

Unit-II

2. Prove that $\{1, \sqrt{2}\cos(k\pi x)\}$, $k = 1, 2, 3, \ldots$ is an orthonormal basis of $L^2([0, 1])$ and its polarity is $(+, +)$.

Or

Prove that $U = F^{-1} AF$ and $U^* = F^{-1} A^*F$. Also prove that

$$(U^*F)(x) = \begin{cases} \overline{s(x)}\, f(x) - s(-x)\, f(-x), & x > 0 \\ s(-x)\, f(x) + \overline{s(x)}\, f(-x), & x < 0 \end{cases}$$

Unit-III

3. If $g \in L^2(R)$, then prove that $\{g(-k) : k \in z\}$ is an orthonormal system if and only if

$$\sum_{k \in Z} |\hat{g}(\xi + 2k\pi)|^2 = 1 \text{ for } a.e.\ \xi \in R.$$

Or

Let ψ be an $L^\infty(R)$ function such that $|\psi(x)| \leq \frac{c}{(1+|x|)^{1+\varepsilon}}$ *a.e.* for some $\varepsilon > 0$.

If $\{\psi_{j,k} : j, k \in Z\}$ is an orthonormal system in $L^2(R)$, then prove that $\int_R \psi(x)\, dx = 0$.

Unit-IV

4. If ψ is a band-limited orthonormal system, then prove that $\sum_{j \in Z} |\hat{\psi}(Z^j \xi)|^2 = 1$ for *a.e.* $\xi \in R - \{0\}$.

Or

Construct a wavelet ψ such that supp. $(\hat{\psi})$ is disjoint from the support of the Fourier transform of the Shannon wavelet.

Unit-V

5. Prove that a function f in $L^2(R)$ belongs to V_0 if and only if $\xi^2 \hat{\phi}(\xi)$ is a 2π periodic function on R.

Or

Prove that the spline wavelets ψ^n, $n = 1, 2, \ldots$ and their associated scalling functions ψ^n have exponential decay at ∞.

SECTION C **10×5=50**

(Long Answer Type Questions)

Note : *Answer the following questions.*

Unit-I

1. Let ψ be such that $\hat{\psi}(\xi) = \chi_1(\xi)$, where $I = [-2\pi, -\pi] \cup [\pi, 2\pi]$. Show that ψ is an orthonormal wavelet for $L^2(R)$.

Or

If $I = [\alpha, \beta]$, then prove that $f \in H_I = P_I(L^2(R))$ iff $f = b_I S$, where $S \in L^2(R)$, b_I is the bell function associated with I, and S is even or odd on $[\alpha - \in, \alpha + \in]$ according to the choice of polarity at α, and even or odd on $[\beta - \in', \beta + \in']$ according to the choice of polarity at β.

Unit-II

2. Prove that the system $\gamma_{j,k}(\xi) = \frac{2^{j/2}}{\sqrt{2\pi}} b(2^j \xi)\, e^{i\frac{2k+1}{2}2^j\xi}$, $j, k \in Z$ is an orthonormal basis for $L^2(R)$, where b restricted to $[0, \infty)$ is a bell function for $[\pi, 2\pi]$ associated with $0 < \varepsilon \leq \frac{\pi}{3}$, $\varepsilon' = 2\varepsilon$, and b is even on R.

Or

Let s satisfy $|s(x)|^2 + |s(-x)|^2 = 1$ for all x with support on $[\varepsilon, \infty)$, and suppose that $s \in C^d$, where C^d is the space of all functions with continuous derivatives up to order d. Then prove that $U_\alpha : C^d \cap L^2(R) \to S_\alpha$ and $U_\alpha^* : S_\alpha \to C^d \cap L^2(R)$, and both operators are one-to-one and onto, where

$S_\alpha = \{f \in C^d(R - \{\alpha\}) \cap L^2(R) : f^{(n)}(\propto \pm)$ exist for $0 \leq n \leq d$,

$$\lim_{x \to \alpha^+} f^{(n)}(x) = 0 \text{ if } n \text{ is odd,}$$

and $\lim_{x \to \alpha^-} f^{(n)}(x) = 0$ if n is even

Unit-III

3. If a scaling function C_p for an MRA has polynomial decay, then prove that the low-pass filter m_0 belongs to $C^\infty(T)$.

Or

Let $\psi \in L^2(R)$ be a compactly supported function such that $\psi \in C^\infty$, then prove that $\{\psi_{j,k} : j, k \in Z\}$ cannot be an orthonormal system in $L^2(R)$, where $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$.

Unit-IV

4. Suppose $f \in L^2(R)$, then prove that f is orthogonal to W_j iff $\sum_{k \in Z} \hat{f}(\xi + 2^{j+1}k\pi)\, \overline{\hat{\psi}(2^{-j}\xi + 2k\pi)} = 0$ for *a.e.* $\xi \in R$.

Or

Suppose $\psi \in L^2(R)$, $b = |\hat{\psi}|$ has support contained in $\left[-\frac{8}{3}\pi, -\frac{2}{3}\pi\right] \cup \left[\frac{2}{3}\pi, \frac{8}{3}\pi\right]$ and ψ is an orthonormal wavelet. Then prove that b is almost everywhere even if and only if $b^2(\xi) + b^2(2\pi - \xi) = 1$ for *a.e.* $\xi \in \left[\frac{2}{3}\pi, \frac{4}{3}\pi\right]$.

Unit-V

5. Prove that a function f in $L^2(R)$ belongs for V_0 iff $\hat{f}(\xi) = \left(\frac{\sin(\xi/2)}{\xi/2}\right)^2 m_f(\xi)$, where m_f is a 2π-periodic function in $L^2(T)$.

Or

For any $n = 1, 2, 3, \ldots.$ prove that the spline wavelet ψ^n satisfies $\sum_{k \in Z} \frac{\hat{\psi}^n(\xi + 2k\pi)}{(\xi + 2k\pi)^{n+1}} = 0$ for *a.e.* $\xi \in R$.

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