G-1/130/21 **Roll No……………..**

M.Sc. I Semester Examination, April-2021

MATHEMATICS

Paper I

(Advanced Abstract Algebra-I)

Time : 3 Hours] [Maximum Marks : 80

Note : *All questions are compulsory. Question Paper comprises of 3 sections.* ***Section A*** *is objective type/Multiple Choice questions with no internal choice.* ***Section B*** *is short answer type with internal choice.* ***Section C*** *is long answer type with internal choice.*

SECTION 'A'

(Objective Type Questions)

Choose the correct answer : **1 × 10 = 10**

1. A group for which $G^{(K)} = (e)$ for some integer K is called :

(a) Nilpotent group (b) P-group

(c) Solvable group (d) None of these

2. Every Abelian group is :

(a) Solvable

(b) not nilpotent

(c) neither nilpotent nor solvable

(d) None of these

3. Polynomial x^2+1 is reducible :

(a) over R (b) over C

(c) over R and C both (d) none of these

4. If $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx_n$EZ[$x$] $n \geq 1$ and P is a prime such that $p^2 \times a_0$., p/a_0, p/a_1, p/a_2, p/a_{n-1}, $b \times a_n$, then,

(a) $f(x)$ is irreducible over Q

(b) $f(x)$ is reducible over Q

(c) $f(x)$ is neither reducible nr irrducible

(d) None of these

5. C be the field of complex number, R be the field of Real Number then G(C/R) is the group of order :

(a) 2 (b) 3

(c) 4 (d) None of these

6. A field which does not possesses proper Algebraic extension is called :

(a) Normal extension (b) Separable extension

(c) Algebraically closed field

(d) None of these

7. An irreducible polynomial $f(x) \in F(x)$ for which all its roots are simple is called :

(a) Separable polynomial

(b) Inseparable polynomial

(c) Menic polynomial

(d) None of these

8. If f be a field of characteristic $\neq$ Z and $x^2 - a \in F(x)$ be an irreducible polynomial over F then Galois group of $x^2 - a$ will be of order :

(a) 1 (b) 2

(c) 3 (d) None of these

Or

If a and b are in K are algebraic over F of degrees m and n respectively then prove $a \neq b$, ab and $ab^{-}(b \neq 0)$ are algebraic over F of degree m and most $mn.2$

3. Define algebraically closed field and prove that A field K is algebraically closed if and only if every non-constant polynomial in K(x) factors in K(x) into linear factors.

Or

Define finite field. If F is a finite field with q elements and $F \subseteq K$, K is also a finite field then prove K has q^n elements where $n = [K : F]$.

4. If K is a field and $\sigma_1, \sigma_2, \ldots\ldots, \sigma_n$ are distinct automorphisms of K, then prove that it is impossible to find elements $a_1, a_2, a_3, \ldots, a_n$ not all zero in K such that

$$a_1\sigma_1(u) + a_2\sigma_2(u) + \ldots\ldots\ldots + a_n\sigma_n(u) = 0, \ \forall u \in K.$$

Or

Define

(i) The Galois group of $f(x)$ over F,

(ii) Galois extension of F,

(iii) In the fundamental theorem of Galois theory prove that if K is a normal extension of F, then $G(K/F) \cong G(E/F) / G(E/K)$.

5. Show that the polynomial $x^5 - 9x + 3$ is not solvable by Radicals.

Or

Prove that all polynomials of degree $n \geq 5$ are not solvable by radicals.

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9. If an irreducible polynomial $p(x) \in F(x)$ over a field F, has a root in a Radical extension of F, then :

(a) $p(x)$ is not solvable (b) $p(x)$ is solvable

(c) $p(x)$ is solvable by Radical over F

(d) None of these

10. If $F = Z/(2)$, then the splitting field of $x^3 + x^2 + 1 \in F(x)$ has :

(a) 6 elements (b) 7 elements

(c) eight elements (d) None of these

SECTION 'B' **5 × 4 = 20**

(Short Answer Type Questions)

Note : *Answer the following questions in 250 words.*

1. Define solvable group and prove that every Abelian group is solvable.

Or

If G is a group and N is a normal subgroup of G such that both G and $\frac{G}{N}$ are solvable, then prove that G is solvable.

2. Let $f(x) \in F(x)$ be a polynomial of degree 2 or 3, then prove that $f(x)$ is reducible if and only if $f(x)$ has a root in F.

Or

Show that every finite extension of a field is algebraic.

3. Show that any field of characteristic zero is perfect.

Or

Prove that every finite field of characteristic P has an automorphism $a \leftrightarrow a^P$.

4. Define fixed field. Let G be a subgroup of the group of all automorphisms of a field K, then prove that fixed field of G is a subfield of K.

Or

Let A(K) be the collection of all automorphism of a field K, then prove that A(K) forms a group with respect to the operation of composite of two functions.

5. Show that it is impossible by straight edge and compass alone to duplicate the cube.

Or

Show that the polynomial $2x^5 - 5x^4 + 5$ is not solvable by Radicals.

SECTION'C' **10 × 5 = 50**

(Long Answer Type Questions)

Note : *Answer the following questions in 500 words.*

1. Define solvable group and prove that a group G is solvable if and only if $G^{(K)} = (e)$ for some integer K.

Or

If G be a Nilpotent group then prove that every subgroup and every homomorphic image of G are Nilpotent.

2. Let E be an extension of a field F. $u \in E$ be algebraic over F and let $p(x) \in F(x)$ be a polynomial of least degree such that $p(u) = 0$, then prove that :

(i) $p(x)$ is irreducible over F,

(ii) If $g(x) \in F(x)$ is such that $g(u) = 0$ then $p(u) \mid g(x)$,

(iii) there is exactly one monic polynomial $p(x) \in F(x)$ is least degree such that $p(u) = 0$.