G-2/231/21 Roll No. ....

M.Sc. II Semester Examination, 2021

MATHEMATICS

Paper II

(Real Analysis-II)

Time : 3 Hours] [Max. Marks : 80

Note : *All questions are compulsory. Question Paper comprises of 3 sections. Section A is objective type/multiple choice questions with no internal choice. Section B is short answer type with internal choice. Section C is long answer type with internal choice.*

SECTION A **1×10=10**

(Objective Type Questions)

Choose the correct answer :

1. Let $f, \alpha : [a, b] \to R$ be bounded functions and α be monotone increasing. If P^* is # a refinement of the partition P of the interval $[a, b]$, then

(a) $L(P, f, \alpha) \leq L(P^*, f, \alpha)$

(b) $L(P, f, \alpha) \geq L(P^*, f, \alpha)$

(c) $L(P, f, \alpha) = L(P^*, f, \alpha)$

(d) None of these

2. If the set of numbers $\Delta\gamma(P)$ is unbounded, then γ is called :

(a) Rectifiable (b) Non-rectifiable

(c) Arc (d) None of these

3. Every enumerable set is measurable and its measure is :

(a) 1 (b) 2

(c) 3 (d) 0

4. Let E be a measurable set, then for any set A :

(a) $m^*(A) = m^*(A \cap E) + m^*(A - [A \cap E])$

(b) $m^*(A) = m^*(A \cap E) + m^*(A + [A \cap E])$

(c) $m^*(A) = m^*(A \cap E) - m^*(A + [A \cap E])$

(d) $m^*(A) = m^*(A \cap E) - m^*(A - [A \cap E])$

5. If f is a real valued function defined on a set E, then its positive and negative parts are defined as :

(a) $f^+ = \max(f, 0)$ and $f^- = \min(f, 0)$

(b) $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$

(c) $f^+ = \min(f, 0)$ and $f^- = \max(f, 0)$

(d) $f^+ = \min(f, 0)$ and $f^- = \min(-f, 0)$

6. Let f be a bounded measurable functions defined on a set E of finite measure and A and B are disjoint measurable subsets of E, then :

(a) $\int_{A+B} f = \int_A f + \int_B f$

(b) $\int_{AB} f = \int_A f + \int_B f$

(c) $\int_{A\cup B} f = \int_A f + \int_B f$

(d) $\int_{A\cup B} f = \int_A f - \int_B f$

7. The Dini derivatives always exists (finite or infinite) for any function f and satisfy :

(a) $D^+ f(x) \geq D_+ f(x)$, $D^- f(x) \geq D f(x)$

(b) $D^+ f(x) \leq D_+ f(x)$, $D^- f(x) \leq D_- f(x)$

(c) $D^+ f(x) \geq D_+ f(x)$, $D^- f(x) \leq D_- f(x)$

(d) $D^+ f(x) \leq D_+ f(x)$, $D^- f(x) \geq D_- f(x)$

8. If f is absolutely continuous on $[a, b]$ and $f = 0$ *a.e.*, then

(a) f is a constant function

(b) f is not a constant function

(c) f vanishes

(d) None of these

9. The exponential function is convex on :

(a) $(-\infty, 0]$ (b) $[0, \infty)$

(c) $(-\infty, \infty)$ (d) $[0, 1]$

10. Every bounded function defined on X is in :

(a) $L^P(X)$ (b) $L^P(\mu)$

(c) $L^4(X)$ (d) $L^\infty(X)$

SECTION B **4×5=20**

(Short Answer Type Questions)

Note : *Attempt one question from each unit.*

Unit-I

1. Let f be a bounded function and α be a monotonically increasing function on $[a, b]$ and if $f \in R(\alpha)$, then for every $\varepsilon > 0$, F a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

Or

Let f be a continuous and α monotonically increasing on $[a, b]$, then $f \in R(\alpha)$ on $[a, b]$.

Unit-II

2. Let A be any set and $E_1, E_2, \ldots\ldots, E_n$ a finite sequence of disjoint measurable sets. Then

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Or

Let $\{f_n\}$ be a sequence of non-negative measurable functions and $f_n \to f$ *a.e.*, on E, then

$$\int_E f \leq \lim_{n\to\infty} \int_E f_n.$$

Unit-III

3. Let (X, B, μ) be a measure space. If $E_i \in B$, $\mu(E_1) < \infty$ and $E_i \supset E_{i+1}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n).$$

Or

Let f be a bounded measurable function defined over a measurable set E, then prove that

$$\left| \int_E f \right| \leq \int_E |f|.$$

Unit-IV

4. A function f is of bounded variation on $[a, b]$ iff f is the difference of two monotone real valued functions on $[a, b]$.

Or

Let f be an integrable function $[a, b]$. If $\int_a^x f(t)\,dt = 0 \ \forall \ x \in [a, b]$, then $f = 0$ *a.e.* in $[a, b]$.

Unit-V

5. Let $0 < p < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $\int_X |fg|\,du \geq \left(\int_X |f|^p\right)^{1/p} \cdot \left(\int_X |g|^q\right)^{1/q}$ provided $\int_X |g|^q \neq 0$.

Or

Let $\{f_n\}$ be a sequence of measurable functions which converge to f *a.e.* on a measurable set E

with $6m(E) < \infty$. Then given $\eta > 0$, there is a set $A \subset$ E with $m(A) < \eta$ such that the sequence $\{f_n\}$ converges to f uniformly on $E - A$.

SECTION C **10×5=50**

(Long Answer Type Questions)

Note : *Attempt one question from each unit.*

Unit-I

1. If $f \in R(\alpha)$ on $[a, b]$ and if $a < c < b$, then $f \in R(\alpha)$ on $[a, c]$ and on $[c, b]$ and

$$\int_a^c f\,dx + \int_c^b f\,dx = \int_a^b f\,dx.$$

Or

Let γ be a continuously differentiable curve on $[a, b]$, then γ is rectifiable and

$$\Lambda_\gamma(a, b) = \int_c^b |\gamma'(t)|\,dt.$$

Unit-II

2. Let E be any set. Then

(a) For any given $\varepsilon > 0$, F an open set $0 \supset E$, such that $m^*(0) < m^*(E) + \varepsilon$, *i.e.*, $m^*(0 - E) < \varepsilon$.

(b) F a G_δ set $G \supset E$, such that $m^*(E) = m^*(G)$.

Or

There exists a non-measurable set in the interval $[0, 1)$.

Unit-III

3. State and prove Lebesgue's Monotone convergence theorem.

Or

$m(\mu)$ is a σ-ring and μ^* is countably additive on $m(\mu)$.

Unit-IV

4. State and prove Lebesgue's Differentiation theorem.

Or

Let E be a set of finite outer measure and I a collection of intervals which cover E in the sense of vitali. Then given $\varepsilon > 0$, F a finite disjoint collection $\{I_1, I_2, \ldots\ldots, I_n\}$ of intervals in I such that

$$m^*\left(E - \bigcup_{i=1}^{n} I_i\right) < \infty.$$

Unit-V

5. If $f, g \in L^2[a, b]$, then $fg \in L^1[a, b]$ and $\| fg \| \leq \|f\|_2 \|g\|_2$.

Or

The L^p-spaces are complete.

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