G-3/332/21 **Roll No……………..**

M.Sc. III Semester Examination, April-2021

MATHEMATICS

Paper III

(Wavelets-I)

Time : 3 Hours] [Maximum Marks : 80

Note : *All questions are compulsory. Question Paper comprises of 3 sections.* ***Section A*** *is objective type/Multiple Choice questions with no internal choice.* ***Section B*** *is short answer type with internal choice.* ***Section C*** *is long answer type with internal choice.*

SECTION 'A'

(Objective Type Questions)

Answer all the questions : **1 × 10 = 10**

1. What is Gabor basis ?

2. Construct the wavelets of Lemarie and Meyer.

3. Prove that $\left\{\sqrt{2}\,\sin\left(\frac{2k+1}{2}\right)\pi x\right\}$, $k = 0, 1, 2\ldots$ is an orthonormal basis of $L^2([0, 1])$.

4. Define the folding operator $F : L^2 (R, C) \to L^2 (R^+, C^2)$.

5. Define MRA with a Riesz basis.

6. Define the low pass filter associated with scaling function ψ.

7. Define band-limited function.

8. What is frame on a Hilbert space ?

9. Define Franklin wavelet.

10. Define splines of order n.

P.T.O.

SECTION 'B' $4 \times 5 = 20$

(Short Answer Type Questions)

Note : *Answer all the following questions.*

1. Give an example of an orthonormal sequence on $T = [-\pi, \pi]$.

Or

Give an example of an orthonormal wavelet on R.

2. Prove that $\{2C^{-}_{j,k} : k \geq 0\}$ is an orthonormal basis for $P^{+-}_{-1j}(L^2(R))$.

Or

Find values of $\lim\limits_{x \to 0^+} f^{(n-k)}(x)$ when $(n-k)$ is even and odd respectively.

3. Let V_j be the space of all functions in $L^2(R)$ which are constant on intervals of the form $(2^{-j} k, 2^{-j}(k+1)]$, $k \in Z$. Then prove that $\{V_j : j \in Z\}$ is an MRA.

Or

Suppose that $g \in L^2(R)$ and that $\{g(-k) : k \in Z\}$ is an orthonormal system, then prove that $|\text{supp}(\hat{g})| \geq 2\pi$.

4. Write necessary and sufficient conditions for the orthonormality of the system $\{\psi_{j,k} ; j, k \in Z\}$.

Or

Suppose that $f \in L^2(R)$ and $\hat{f}$ has a support contained in $I = (a, b)$, where $b - a \leq 2^{-J}\pi$ and $I \cap [-\pi, \pi] = \phi$, then prove that for all $j \in Z$,

$$(Q_j f)^\wedge(\xi) = f(\xi) |\hat{\psi}(2^{-J}\xi)|^2 \text{ a.e. on I.}$$

5. If $f \in L^2(T)$, then prove that $\{f, Uf, \ldots\ldots, U^N f\}$, where $U \equiv U_j$ is an orthonormal system if and only if $\sum\limits_{l \in Z} |F[f](n+2^j l)|^2 = 2^{-j}$ for $n = 0, 1, \ldots\ldots, N = 2^j - 1$.

Or

Prove that $f \in C_j^{(m)}$ if and only if $f \in B_{j+1}^{(m)}$, $F[f](0) = 0$ and $\sum\limits_{l \in Z} \dfrac{F[f](n+2^i l)}{(n+2^j l)^{m+1}} = 0$, for $n = 1, 2; \ldots\ldots\ldots, N = 2^j - 1$.

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5. Prove that the basic spline of order n, Δ^n, satisfies following property :

$$\hat{\Delta}^n(\xi) = e^{-i\frac{n+1}{2}\xi}\left[\frac{\sin\left(\frac{\xi}{2}\right)}{\xi/2}\right]^{n+1}$$

Or

If Cj is the orthogonal complement of B_j in B_{j+1}, then prove that there exists a $g_j \in C_j$ such that $\{g_j, Ug_j^N\}$ is an orthonormal basis for C_j, $j = 0, 1, 2, \ldots\ldots$

SECTION'C' 10 × 5 = 50

(Long Answer Type Questions)

Note : *Answer the following question.*

1. State and prove Balian-Low theorem.

Or

Let $I = [\alpha, \beta]$; then show that $f \in H_1 = P_1(L^2(R))$ if and only if $f = b_1 S$, where $S \in L^2(R)$, b_1 is the bell function associated with I and S is even or odd on $[\alpha-\varepsilon, \alpha+\varepsilon]$ according to the choice of polarity at α, and even or odd on $[\beta - \varepsilon', \beta + \varepsilon']$ according to the choice of polarity at β.

2. If the polarities are $(-, -)$, $(+, -)$ and $(+, +)$ at (α, β), then prove that the same is true for :

(i) $\left\{\sqrt{\frac{2}{|I|}} b_1(x)\cos\left(\frac{2k+1}{2}\frac{\pi}{|I|}(x-\alpha)\right)\right\}$, $k = 0, 1, 2, \ldots\ldots$

(ii) $\left\{\sqrt{\frac{1}{|I|}} b_1(x), \sqrt{\frac{2}{|I|}} b_1(x)\cos(k\frac{\pi}{|I|}(x-\alpha)\right\}$, $k = 1, 2, 3, \ldots\ldots$

Or

If $U = F^{-1}AF$ and $U^{\infty} = F^{-1}A{*}F$, then show that U is unitary. Also prove that :

$$(U^*f)(x) = \begin{cases} \overline{S(x)}.f(x) - S(-x)f(-x), x > 0; \\ S(-x)f(x) + \overline{S(x)}f(-x), x < 0; \end{cases}$$

and $U^{\infty} X_{[0,\infty]} U = P^{+}_{0,\varepsilon}$, where $P^{+}_{0,\varepsilon}$ has its usual meaning.

3. Let ψ be on $L^{\infty}(R)$ function such that :

$$|\psi(x)| \le \frac{C}{(1+|x|)^{1+\varepsilon}} \text{ a.e. for some } \varepsilon > 0.$$

If $\{\psi_{j,k} : j, k \in Z\}$ is an orthonormal system in $L^2(R)$, then prove that $\int_R \psi(x)\, dx = 0$.

Or

For any integer $r = 0, 1, 2, \ldots\ldots$ there exists an orthonormal wavelet ψ with compact support such that ψ has bounded derivatives upto order r. Prove it.

4. If ψ is a band-limited orthonormal wavelet, then prove that $\sum_{j\in Z} |\hat{\psi}(2^i\xi)|^2 = 1$ for a.e. $\xi \in R-\{0\}$.

Or

If ψ is a band-limited orthonormal wavelet such that $|\hat{\psi}|$ is continuous at 0, then prove that $|\hat{\psi}| = 0$ a.e. in an open neighbourhood of the origin.