G-4/430/21

Roll No. ....

M.Sc. IV Semester Examination, 2021

MATHEMATICS

Paper III

(Wavelets–II)

Time : 3 Hours] [Max. Marks : 80

Note : *All questions are compulsory. Question Paper comprises of 3 sections. Section A is objective type/multiple choice questions with no internal choice. Section B is short answer type with internal choice. Section C is long answer type with internal choice.*

SECTION A 1×10=10

(Objective Type Questions)

1. "Every inner product space is a normed linear space". (True/False)

2. If $\psi_{j,k}(x) = 2^{j/2}\ \psi(2^j x - k)$, then $(\psi_{j,k})\ \hat{}\ (\zeta)$ = ...

3. Give an example of dense subset $\mathcal{D}$ of $L^2(\mathbb{R})$.

4. Define Minimally supported frequency wavelets.

5. Define Journe Wavelet.

6. A wavelet $\psi \in L^2(\mathbb{R})$ is an MRA wavelet if and only if for $\zeta \in \mathbb{R}$.

P.T.O.

(a) $D_\psi(\xi) = 0$ (b) $|D_\psi(\xi)| = 1$

(c) $D_\psi(\xi) = 1$ (d) None of the above.

7. Define frame operator.

8. Define Zak transform.

9. Define $\overline{|N}$, used in the definition of Discrete Fourier Transform.

10. What do you know about Fast Fourier Tranform ?

SECTION B 4×5=20

(Short Answer Type Questions)

Note : Attempt one questions from each unit with internal choice.

Unit-I

1. Let H be a Hilbert space and $\{e_j : j = 1, 2, ...\}$ be a family of elements of H. Then prove that

(i) $\|f\|^2 = \sum_{j=1}^{\infty} |\langle f, e_j \rangle|^2$ holds for all $f \in H$ if and only if,

(ii) $f = \sum_{j=1}^{\infty} \langle f, e_j \rangle e_j$, with convergence in H, for all $f \in H$.

Or

Suppose $\{e_j : j = 1, 2, ...\}$ is a family of elements in a Hilbert space H such that

$$\| f \|^2 = \sum_{j=1}^{\infty} |\langle f, e_j \rangle|^2$$

holds for all f belonging to a dense subset D of H, when prove that this above equality is valid for all $f \in H$.

Unit-II

2. Let $\{v_j : j \geq 1\}$ be a family of vectors in a Hilbert space H, then prove that the series

$v_n = \sum_{m=1}^{\infty} \langle v_n, v_n \rangle v_m$, for all $n \geq 1$ is a well defined element of H.

Or

Let $\mu_1, ..., \mu_n$ be 2π-periodic functions and set

$$M_j = \sup_{\xi \in T} (|u_j(\xi)|^2 + |\mu_j(\xi + \pi)|^2).$$

Then prove that

$$\int_{-2^n \pi}^{2^n \pi} \prod_{j=1}^{n} |\mu_j(2^{-j}\xi)|^2 \, d\xi \leq 2\pi M_1 .... M_n.$$

Unit-III

3. Prove that $F^* F$ commutes with translation by integers and with integral modulations.

Or

If $\left\{\tilde{\phi}_j \equiv s^{-1}(\phi_j), j \in J\right\}$ is a dual frame, then prove that

$$\sum_{j \in J} |\langle f_1 \tilde{\phi}_j \rangle|^2 = \langle S^{-1} f, f \rangle \text{ for all } f \in H.$$

Unit-IV

4. If $E_k(N) = v^k$ with $v \in \overline{N}$, then prove that

$$\left\{\frac{1}{\sqrt{N}} E_k : k = 0, 1, 2, ..., N-1\right\}$$

is an orthonormal basis for $l^2(\overline{N})$.

Or

What do you mean by Fast Fourier Transform and its use.

Unit-V

5. Define windows functions used to express discrete version of the local sine and cosine bases.

Or

What do you understand by Decomposition of wavelets.

SECTION C 10×5=50

(Long Answer Type Questions)

Note : Attempt all questions.

Unit-I

1. Suppose $0 < a < b < \infty$, $f \in D$ with

supp $(\hat{f}) \subseteq \{\xi : a < |\xi| < b\}$ and δ = diam (supp $(\hat{f})$). Then prove that

$$\sigma(\xi) = \sum_{j \in Z} \sum_{k \neq 0} 2^{-j} \left| \hat{f}(e^{-j}\xi) \right| . \left| \hat{f}(2^{-j}(\xi + 2k\pi) \right|$$

$$\leq \frac{\delta}{\pi}\left(1 + \log_2 \frac{b}{a}\right) \| \hat{f} \|_{\infty}^{2} \text{ for all } \xi \in \mathbb{R}.$$

Or

Let $\psi \in L^2(\mathbb{R})$ be such that $|\hat{\psi}| = \chi_K$ for a measurable set $K \subseteq R$. Then prove that ψ is a wavelet if and only if there exist a partition $\{I_l : l \in Z\}$ of I, a partition $\{K_i : l \in Z\}$ of K and two integer valued sequences $\{j_l : l \in z\}$, $\{k_i : l$ Î $Z\}$ such that

(i) $K_l = 2^{jl} I_l \; l \in Z$, and

(ii) $\{K_l + 2k_l\pi : l \in Z\}$ is a partition of I.

Unit-II

2. If ψ is an orthonormal wavelet, then prove that

$$\hat{\psi}(2^n \xi) = \sum_{j=1}^{\infty} \sum_{k \in Z} \hat{\psi}(2^2(\xi + 2k\pi) \; \overline{\hat{\psi}(2^j(\xi + 2k\pi))}$$

$$\hat{\psi}(2^j \xi) \text{ for all } n \geq 1.$$

Or

For any orthonormal wavelet $\psi \in L^2(\mathbb{R})$ prove that ψ is an MRA if and only if dim $F_\psi(\xi) = 1$ for $\xi \in \mathbb{T}$.

Unit-III

3. State and prove Balian Law theorem for frames.

Or

Suppose that $g \in L^2(\mathbb{R})$ and that

$\{g_{m,n}(x) = e^{2\pi imx} g(x - n) : m, n \in Z\}$

is a frame with frame bounds A and B.

Then prove that

$0 < A \leq | \; Rg(s, t)|^2 \leq B \infty$ on $\mathbb{T}^2$,

and $(R\tilde{g}_{m,n})(s,t) = \dfrac{e^{2\pi ims} e^{2\pi int}}{\overline{Rg(s,t)}}$.

Unit-IV

4. Explain in detail what do you mean by Discrete Fourier transform and Fast Fourier transform. Also pvove that if P, Q and T all three $n \times n$ matrices such that $PQ = T$, then $P^\# \, Q^d = T^\#$.

Or

Prove that the vectors

$$\left\{\sqrt{\frac{2}{N}}C_k(N) : k = 0,1,\ldots,N-1\right\}$$

forms an orthonormal basis for R^N.

Unit-V

5. Prove that the sequence $\{u_{j,k} : j \in Z, 0 \le k \le l_j - 1\}$ given by

$$u_{j,k}(x) = \sqrt{\frac{2}{l_j}}w_{j(x)\cos}\left(\pi\left(k+\frac{1}{2}\right)\left(\frac{x-a_j}{l_j}\right)\right), x \in Z$$

is an orthonormal system for $l^2(z)$.

Or

Explain in detail the reconstruction algorithm for wavelets.

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