Topological Space

Let X be a non empty set and let T be a collection (class) of subsets of X, Then T is called a **topology** for X (or topology on X) if it satisfies the following axioms :

[T_1]- X and ϕ belong to T

i.e $X \in T$, $\phi \in T$

[T_2]- The intersection of any two sets in T belongs to T

i.e if $G_1 \in T, G_1 \in T$

then $G_1 \cap G_2 \in T$

[T_3]- The union of any number of sets(arbitrary collection of sets) in T belongs to T.

i.e if $G_\lambda \in T$ $\forall \lambda \in \Lambda$ where Λ is an arbitrary(Index) set

then $\cup\{G_\lambda : \lambda \in \Lambda\} \in T$

The Pair (X, T) is called a **topological space**

Notes – When (X, T) is a topological space. Elements of X are called **points of the space** and the members of topology T are called ***T*- open sets** (or simply **open sets**)

Topological Space (In terms of open sets)

Let X be a non-empty set and T be collection of subsets of X (called open sets), satisfying following axioms

T_1 : The empty set and the whole space are open.

T_2 : The intersection of two open sets is open.

T_3 : The Union of arbitrary collection of open sets is open.

Then T is called a topology on set X, and the pair (X, T) is called a **topological space.**

Examples

1. Let X = {a,b, c,}

   then T = {∅, {b}, {a,b} X} is a topology on X, since all three axioms of being topology are satisfied as follows:

[T_1] - ∵ ∅ ∈ T, X ∈ T

So T_1 is satisfied

[T_2] - ∵ ∅ ∩ {b} = {b} ∈ T

{b} ∩ {a,b} = {b} ∈ T

{a,b} ∩ X = {a, b} ∈ T etc.

So we conclude that ∀ G_1, G_2 ∈ T

$G_1, G_2 \in T \Rightarrow G_1 \cap G_2 \in T$

Therefore T_2 is satisfied

[T_3] - ∵ ∅ ∪ {b} = {b} ∈ T

$\emptyset \cup \{a, b\} = \{a, b\} \in T$

$\{b\} \cup \{a, b\} = \{a, b\} \in T$

$\emptyset \cup \{b\} \cup \{a, b\} \cup X = X \in T$ etc.

Thus union of arbitrary collection of members of T is also a member of T.

∴ T_3 is satisfied

2. Let X be a non-empty set and T_x Consists of X and all those subsets of X, which do not contain a particular point $x \in X$. Show that T_x is a topology on X.

Solution - Here $X \neq \emptyset$

T_x = Collection of set X, and the subsets of X not containing point $x \in X$

$= \{X\} \cup \{G : G \subset X \text{ and } x \notin G\}$

To show that T_x is a topology on X.

[T_1] - ∵ $x \notin \emptyset$

∴ $\emptyset \in T_x$

Also $X \in T_x$ [by definition of T_x]

∴ T_1 is satisfied

[T_2] - Let $G_1, G_2 \in T_x$ then

$G_1, G_2 \in T_x \Rightarrow x \notin G_1,\ x \notin G_2$

$\Rightarrow x \notin G_1 \cap G_2$

$\Rightarrow G_1 \cap G_2 \in T_x$ $\quad\therefore$ T_2 is satisfied

[T_3] - Let $G_\lambda \in T_x \quad \forall \lambda \in \Lambda$ then

$G_\lambda \in T_x \ \forall \lambda \in \Lambda \Rightarrow x \notin G_\lambda \quad \forall \lambda \in \Lambda$

$\Rightarrow x \notin \cup \{G_\lambda : \lambda \in \Lambda\}$

$\Rightarrow \cup \{G_\lambda : \lambda \in \Lambda\} \in T_x$

$\therefore$ T_3 is satisfied

Hence T_x is a topology on X proved

3. Let X be a non-empty set and T be the collection of all those subsets of X whose complements are finite, together with ∅, show that T is a topology on X.

Solution - $Here\ X \neq \emptyset \ \ and$

T = Collection of all those subsets of X whose complements are finite, together with ∅

$= \{\emptyset\} \cup \{G : G \subset X \ and\ G'\ is\ finite\}$

To show that T is a topology on set X.

[T_1] : $\because \quad X' = \emptyset$ (a finite set)

$\Longrightarrow X \in T$

Also by definition of T, $\emptyset \in T$.

$\therefore$ T_1 is satisfied

[T_2] : Let $G_1, G_2 \in$ T then

$G_1, G_2 \in T \Longrightarrow G'_1$ & G'_2 are finite

$\Longrightarrow G'_1 \cup G'_2$ is finite

$\Longrightarrow (G_1 \cap G_2)'$ is finite [by De Morgan's Law]

$\Longrightarrow G_1 \cap G_2 \in T$

$\therefore$ T_2 is satisfied

[T_3] : Let $G_\lambda \in T \quad \forall \lambda \in \Lambda$ then

$G_\lambda \in T \implies G'_\lambda$ is finite $\forall\ \lambda \in \Lambda$

$\implies \cap \{ G'_\lambda : \lambda \in \Lambda \}$ is finite

$\implies [\cup \{ G_\lambda : \lambda \in \Lambda \}]'$ is finite

[by De Morgan's law]

$\implies \cup \{ G_\lambda : \lambda \in \Lambda \} \in T$

$\therefore T_3$ is satisfied

4. Let R be the set of real numbers and U be the class of all open sets of real number together with $\emptyset$, show that U is a topology on set R

sol^n : **Recall**- G is open set in R $\Rightarrow \forall\ x \in G, \exists\ \varepsilon > o$ such that $(x-\varepsilon,\ x+\varepsilon) \subset G$

Here R = set of all real numbers.

U = { $\emptyset$ } U {G : G is open in R }

To show that U is a topology on R

$[T_1]$: $\emptyset \in U$ [Given]

$R \in U$ [$\because$ R is open set in R because for any $x \in R$,

$\exists\ \varepsilon > o$ s.t $(x-\varepsilon, x+\varepsilon) \subset R$]

$\therefore T_1$ is satisfied

$[T_2]$: Let $G_1, G_2 \in U$ then either $G_1 \cap G_2 = \emptyset$ or $G_1 \cap G_2 \neq \emptyset$

When $G_1 \cap G_2 = \emptyset$ $G_1, G_2 \in U \implies G_1 \cap G_2 (= \emptyset) \in U$

$\therefore T_2$ is satisfied

When $G_1 \cap G_2 \neq \emptyset$ let $x \in G_1 \cap G_2$

Then $x \in G_1$ *and* $x \in G_2$

$\because$ G_1 is open $\exists\ \varepsilon_1 > o$ s. t $(x - \varepsilon_1,\ x + \varepsilon_1) \subset G_1$

Also G_2 is open $\exists\ \varepsilon_2 > o$ s. t $(x - \varepsilon_2,\ x + \varepsilon_2) \subset G_2$

Taking $\varepsilon = \min \{ \varepsilon_1, \varepsilon_2 \}$,

we have $\varepsilon > 0$, such that

$(x-\varepsilon, x+\varepsilon) \subset (x-\varepsilon_1\ \ x+\varepsilon_1) \subset G_1$

And $(x-\varepsilon, x+\varepsilon) \subset (x-\varepsilon_2\ \ x+\varepsilon_2) \subset G_2$

Or $(x-\varepsilon, x+\varepsilon) \subset G_1 \cap G_2$

$\Rightarrow G_1 \cap G_2 \in \mathcal{U}$

[T_3] : Let $\{ G_\lambda : \lambda \in \Lambda \}$ be an arbitray collection of sets of $\mathcal{U}$ and

$x \in \cup\{ G_\lambda : \lambda \in \Lambda \}$

Then $x \in G_\lambda$ for some λ

$\because$ $G_\lambda \in \mathcal{U}$ $\exists\, \varepsilon > o$ s. t

$(x - \varepsilon, x + \varepsilon) \subset G_\lambda$ for some λ

$\Rightarrow (x - \varepsilon, x + \varepsilon) \subset G_\lambda \subset \cup\{ G_\lambda : \lambda \in \Lambda \}$

$\Rightarrow \cup\{ G_\lambda : \lambda \in \Lambda \} \in \mathcal{U}$

$\therefore$ T_3 is satisfied

Hence U is a topology on set R.

5. Let X be a non-empty uncountable set and let T be the collection consisting of the empty set and all those subsets of X whose complements are countable show that T is a topology on X.

Proof - Let X be a non-empty uncountable set

$T = \{\emptyset\} \cup \{ G : G \subset X, G' = X - G \text{ is countable} \}$

Thus $G \in T \Rightarrow G'$ is countable

$\Rightarrow G'$ is finite or enumerable

To show that T is a topology on X :

[T_1] : $\emptyset \in T$ [by assumption]

$\because$ X' = X-X

$= \emptyset$ (a finite set), which is countable

$\therefore X \in T$

Thus T_1 is satisfied

[T_2] : Let $G_1, G_2 \in T$ then

$G_1, G_2 \in T \Rightarrow G'_1, G'_2$ are countable

$\Rightarrow G'_1 \cup G'_2$ is countable

$\Rightarrow (G_1 \cap G_2)'$ is countable [by De Morgan's law]

$\Rightarrow G_1 \cap G_2 \in T$

$\therefore$ T_2 is satisfied

[T_3] : Let $G_\lambda \in T \;\; \forall \lambda \in T$ then

$G_\lambda \in T, \; \forall \lambda \in \Lambda \;\Rightarrow\; G'_\lambda$ is countable , $\forall \lambda \in \Lambda$

$\Rightarrow \; \bigcap_{\lambda \in \Lambda} G_\lambda{}'$ is countable

$\Rightarrow \; (\bigcup_{\lambda \in \Lambda} G_\lambda)'$ is countable [by De Morgan's Law]

$\Rightarrow \; \bigcup_{\lambda \in \Lambda} G_\lambda \in T$

$\therefore$ T_3 is satisfied

These arguments prove that T is a Topology on X.

---

6. Suppose T is a family, consisting of $\emptyset$ and all those subsets A_n of N such that

$A_n = \{n, n+1, n+2 \ldots..\} \;\; \forall n \in N$

(i) Show that T is a topology on N.

(ii) Find the open sets containing the numbers 2 and 7 respectively

(iii) What are open sets containing 5 ?

Sol^n- N = The set of all natural numbers

$A_n = \{n, n+1, n+2 \ldots.\} \; \forall n \in N$

$T = \{\emptyset\} \cup \{A_n : n \in N\}$

(i) To show that T is a topology on N-

[T_1] - $\emptyset \in T$ (Given)

$A_1 = \{1, 2, 3 \ldots\ldots..\}$

$= N$

$\therefore A_1 \in T \;\; \Rightarrow N \in T$

Thus T_1 is satisfied

[T_2] – Let $A_n, A_m \in T$ then

$A_n, A_m \in T$

$\Rightarrow A_n \subset A_m$ (if m < n) or $A_m \subset A_n$, (if n < m)

$\Rightarrow A_n \cap A_m = A_n$ (if m < n) or $A_n \cap A_m = A_m$ (if n < m)

$\Rightarrow A_n \cap A_m \in T$ (either n < m or m < n)

Thus T_2 is satisfied

[T_3] – Let $A_i \in T \quad \forall i \in \Lambda \subset N$

Suppose n_0 be the smallest positive integer (ie index) Contained in Λ then

$\cup\{A_i : i \in \Lambda\} = A_{n_0} \cup A_{n_0+1} \cup A_{n_0+2} \cup$ ……………..

$= A_{n_0}$ [$\because n_0$ is the smallest $\Rightarrow A_{n_0}$ is superset of all other sets $A_{n_0+1}, A_{n_0+2}, A_{n_0+3}$ …………]

$\therefore$ $A_i \in T$ $\forall i \in \Lambda$

$\Rightarrow \cup\{ A_i : I \in \Lambda \} \in T$ [$\because$ $An_0 \in T$]

Thus T_3 is satisfied ,hence T is a topology on N.

(II) $\because$ A_n = {n, n+1, n+2 ……..}

Open sets containing 2 are

A_1 = { 1, 2, 3 ……..}

And A_2 = {2, 3, 4 ……..}

Open sets containg 7 are

A_1 = { 1, 2, 3 ……..}, A_2 = {2, 3, 4 ……..}, A_3 = {3, 4, 5..},

A_4, A_5 , A_6, and A_7 ={7, 8, 9 …….}
