Some Kinds of Topology and Topological space

Discrete Topological Space – Let D be the collection of all subsets of a non-empty set X, then D is a topology for X called the discrete topology. The pair (X, D) is called a discrete topological space.

Example - Let $X = \{\mathbf{1,2,3}\}$

$$D = \text{collection of all subsets of}$$

$$= \{X, \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}. \{2,3\}\}$$

To show that: D is a topology on X

[T_1]: $X \in D, \emptyset \in D$

[T_2]: $\because X \cap \emptyset = \emptyset \in D$

$$\{1\} \cap \{2\} = \emptyset \in D$$

$$\{3\} \cap \{1,2\} = \emptyset \in D$$

$$\{1,3\} \cap \{2,3\} = \{3\} \in D \quad \text{...etc.}$$

Thus intersection of any two subsets of D belongs to D,

i.e. $G_{1,}G_2 \in D \Rightarrow G_1 \cap G_2 \in D \quad \forall G_{1,}G_2 \in D$

[T_3]: $\because X \cup \emptyset = X \in D$

$$\{1\} \cup \{2\} = \{1,2\} \in D$$

$$\{3\} \cup \{1,2\} = X \in D$$

$$\{1\} \cup \{2\} \cup \{1,3\} \cup \{2,3\} = X \in D \text{... etc.}$$

Thus union of any number of subsets of D belongs to D,

i.e. $G_\lambda \in D, \ \forall \lambda \in \Lambda \Rightarrow \bigcup G_\lambda \in D$

hence D is a topology on X,called **discrete topology** on X.

Indiscrete Topological space – Let X be any non-empty set then collection I = $\{\emptyset, X\}$ consisting of the empty set and the whole space is always a topology for X called the indiscrete (trivial) topology. The pair (X, I) is called on Indiscrete topological space.

Example - Let $X = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$

$$I = \{X, \emptyset, \}$$

To show that: I is a topology on X

[T_1]: $X \in I, \emptyset \in I$

[T_2]: $\because X \cap \emptyset = \emptyset \in I$

$$\phi \cap X = \emptyset \in I$$

.Thus intersection of any two subsets of I belongs to I,

i.e. $G_{1,}G_2 \in I \Rightarrow G_1 \cap G_2 \in I \quad \forall G_{1,}G_2 \in I$

[T_3]: $\because X \cup \emptyset = X \in I$

Thus union of any number of subsets of I belongs to I,

i.e. $G_\lambda \in I, \ \forall \lambda \in \Lambda \Rightarrow \bigcup G_\lambda \in I$

hence$I$ is a topology on X,called **indiscrete topology** on X.

The Usual Topological space - Let R be the set of all real numbers and U consists of $\emptyset$ and all those subjects G of R having the property that, to each x ∈ G, $\exists \ \varepsilon > o$ such that $(x - \varepsilon, x + \varepsilon) \subset$ G. Then U is a topology for R called the usual topology (standard topology or Euclidean topology) for R and the pair (R, U) is called the usual topological space.

Co-finite topology :- Let X be any set and let T be the collection of all those subsets of X whose complements are finite, Then T is a topology for X called the co-finite topology (or the finite- complement topology)and the pair (X, T) is called cofinite topological space.

Co-countable topology - Let X be a non empty set and let T consists of all those subsets of X whose complements are countable sets, together with the empty set, then T is a topology for X called the co-countable topology

Lower limit topology (Right half open interval topology) Let R be the set of real number and let S consists of subsets of R defined as follows:

(i) $\emptyset \in S$

(ii) a non-empty subset G of R belongs to S iff to each p ∈ G, then exists a right half open interval [a,b) where a, b ∈ R, a < b such that p ∈ [a,b) ⊂ G, then S is a topology for set R called the lower limit topology (Right half open interval topology) for R,

Left Ray topology for R – Let R be the set of all real numbers, for each a ∈ R define :

$$La = \{x \in R : x < a\}$$ to be an open left ray of real numbers.

The point a is called right end point of La.

Let T consists of all possible left rays together with ∅ and R, then T is a topology for R called the left ray topology.

Comparison of Topologies
Let X be a non-empty set and T_1 and T_2 are two topologies on set X, then (i) If $T_1 \subset T_2$ T_1 is called coarser, weaker or smaller topology than T_2 and T_2 is called finer ,stronger or larger than T_1 And T_1 and T_2 are said to be comparable (ii) If $T_1 \not\subset T_2$ and $T_2 \not\subset T_1$ T_1 and T_2 are said to be uncomparable

Intersection and Union of topologies

Theorem – If T_1 and T_2 are two topologies on a non-empty set X, then their inter-section $T_1 \cap T_2$ is also a topology for X

Proof –T_1 and T_2 are two topologies on set X.

To show that : $T_1 \cap T_2$ is a topology on set X

[T_1] - $\because \emptyset \in T_1, \emptyset \in T_2$ [$\because T_1$ and T_2 are topologies]

$\therefore \emptyset \in T_1 \cap T_2$

Also $X \in T_1$ and $X \in T_2$

$\therefore X \in T_1 \cap T_2$

Thus $\emptyset, X \in T_1 \cap T_2$

So T_1 is satisfied.

[T_2] – Let $G_1, G_2 \in T_1 \cap T_2$

Then $G_1 \in T_1 \cap T_2 \Rightarrow G_1 \in T_1$ and $G_1 \in T_2$

$G_2 \in T_1 \cap T_2 \Rightarrow G_2 \in T_1$ and $G_2 \in T_2$

Now $G_1 \in T_1$ and $G_2 \in T_1 \Rightarrow G_1 \cap G_2 \in T_1$

And $G_1 \in T_2$ and $G_2 \in T_2 \Rightarrow G_1 \cap G_2 \in T_2$

Hence $G_1 \cap G_2 \in T_1 \cap T_2$

$\therefore T_2$ is satisfied.

[T_3] – Let $\{G_\lambda : \lambda \in \Lambda\} \in T_1 \cap T_2$

Then $G_\lambda \in T_1 \cap T_2$, $\forall \lambda \in \Lambda$

$\Longrightarrow G_\lambda \in T_1$ and $G_\lambda \in T_2 \quad \forall \lambda \in \Lambda$

Now $G_\lambda \in T_1 \; \forall \lambda \in \Lambda$

$\Longrightarrow \bigcup_{\lambda \in \Lambda} G_\lambda \in T_1$ ....(i) [$\because T_1$ is a topology,by T_3]

Also $G_\lambda \in T_2 \; \forall \lambda \in \Lambda$

$\Longrightarrow \bigcup_{\lambda \in \Lambda} G_\lambda \in T_2$ ....(ii) [$\because T_2$ is a topology, by T_3]

$\therefore$ From (i) and (ii)

$\bigcup_{\lambda \in \Lambda} G_\lambda \in T_1 \cap T_2$

Thus T_3 is satisfied

Hence $T_1 \cap T_2$ is also a topology for X

Intersection of arbitrary collection of topologies

TH- If $\{T_\lambda : \lambda \in \Lambda\}$ where Λ is an arbitrary indexing set, be a collection of topologies, for X then $\bigcap_{\lambda \in \Lambda} T_\lambda$ is also a topology for X.

Proof – [T_1] - $\because T_\lambda$ is a topology for X, $\forall \lambda \in \Lambda$

$\therefore X \in T_\lambda$ and $\emptyset \in T_\lambda \ \forall \lambda \in \Lambda$

$\Rightarrow X \in \bigcap_{\lambda \in \Lambda} T_\lambda$ and $\emptyset \in \bigcap_{\lambda \in \Lambda} T_\lambda$

$\therefore T_1$ is satisfied

[T_2] – Let $G_1, G_2 \in \bigcap_{\lambda \in \Lambda} T_\lambda$

Then $G_1 \in \bigcap_{\lambda \in \Lambda} T_\lambda$

$\Rightarrow G_1 \in T_\lambda$ for each $\lambda \in \Lambda$ …………….(i)

And $G_2 \in \bigcap_{\lambda \in \Lambda} T_\lambda$

$\Rightarrow G_2 \in T_\lambda$ for each $\lambda \in \Lambda$ …………….(ii)

$\therefore$ form (i) and (ii) we have

$G_1 \cap G_2 \in T_\lambda \quad \forall \lambda \in \Lambda$

$\Rightarrow G_1 \cap G_2 \in \bigcap_{\lambda \in \Lambda} T_\lambda$ [$\because$ Each T_λ is a topology by T_2]

Thus T_2 is satisfied.

[T_3]- Let $G_\alpha \in \bigcap_{\lambda \in \Lambda} T_\lambda$ foe each $\alpha \in \triangle$ where $\triangle \equiv$ an arbitrary index set

Then $G_\alpha \in T_\lambda$ for each $\alpha \in \triangle$ and $\forall \lambda \in \Lambda$

$\Rightarrow \bigcup_{\alpha \in \triangle} G_\alpha \in T_\lambda \quad \forall \lambda \in \Lambda$ [$\because T_\lambda$ is a topology]

$\Rightarrow \bigcup_{\alpha \in \triangle} G_\alpha \in \bigcap_{\lambda \in \Lambda} T_\lambda$

$\therefore T_3$ is satisfied.

Hence $\bigcap_{\lambda \in \Lambda} T_\lambda$ is also a topology for x ….. H.P.

Union of Topologies is not necessarily a topology – Let X = {a,b,c} T_1 = {X, $\emptyset$, {a}} and T_2 = {X, $\emptyset$, {b}} are two topologies on X, Then $T_1 \cup T_2$ = {X, $\emptyset$, {a},{b}} obviously $T_1 \cup T_2$ is not a topology on X.