Open set in a topological space

Let (X, T) be a topological space, then the members of T are said to be open sets or T -open set, i.e set $A \subset X$,is said to be T-open if $A \in T$.

Definition of Topological space in terms of open sets

Let X be a non-empty set and T be collection of subsets of X (called open sets), satisfying following axioms

T_1 : The empty set and the whole space are open.

T_2 : The intersection of two open sets is open.

T_3 : The Union of arbitrary collection of open sets is open.

Then T is called a **topology** for X and the Pair (X, T) is called a **topological space**

Examples of open sets

1. $\emptyset$ and X are open sets.

2. If $X = \{a, b, c, d, e\}$

And $T = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

then T is a topology on X.

$$\therefore \quad T - open\ sets\ are\ X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}$$

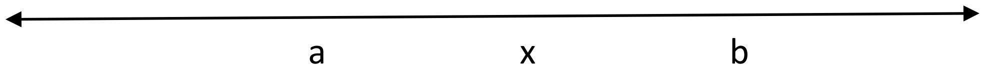
3. Prove that every open interval on R is a U-open set.

Proof – ***U*-open set(definition)**- Let (R, U) be the usual topological space,then the members of U are called U-open sets

or

$G \subseteq R$ is said to be U-open if $\forall x \in G, \exists \varepsilon > 0\ s.t. (x - \varepsilon, x + \varepsilon) \subset G$.

Let (a, b) be any open interval on R, $a, b\ (a < b) \in R$ and $x \in (a, b)$ then



Taking $\varepsilon = \min\{x-a, b-x\}$ it is evident that

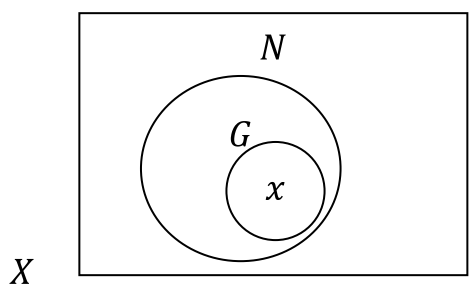
$$(x-\varepsilon, x+\varepsilon) \subset (a,b)$$

$\therefore$ (a,b) is a U-open set.

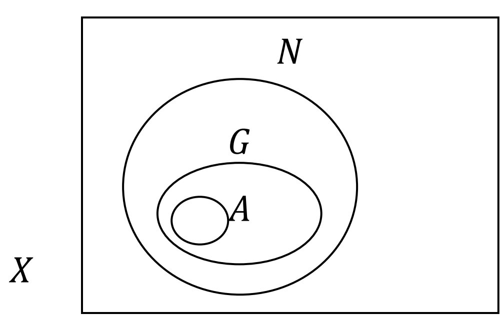
Neighbourhood

Let (X,T) be a topological space and let $x \in X$, then a subet N of X is called a T-neighbourhood of x if $\exists$ a T-open set G such that $x \in G \subset N$.Thus

T-neighbourhood of $x =$ superset $N(N \subset X)$ of a T-open set G containing x



Similarly a subet N of X is called a T-neighbourhood of $A \subset X$ iff $\exists$ a T-open set G such that $A \subset G \subset N$. Thus N is a nbd of every point of A.



Note –1. Nbd of any point $x \in X$ can be open or closed or both.

2.If $x \in G$,where G is an open set, $G - \{x\}$ is said to be **deleted nbd of** x.

Neighbourhood system

Let (X, T) be a topological space and let $x \in X$, then the collection of all $T - nbd$ of x is called neighbourhood system of x,denoted by $\boldsymbol{N_x}$ ***or*** $\boldsymbol{N(x)}$

Example 1. If $X = \{a, b, c, d, \}$

And $T = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, d\}\}$ be a topology then find $\boldsymbol{N_a, N_b}$

To find neighbourhood system of $\boldsymbol{a}$:

T-open sets containing point a are : $X, \{a\}, \{a, b\}, \{a, b, d\}$

And superset of X is X

supersets of $\{a\}$ are: $\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X$

superset of $\{a, b\}$ are: $\{a, b\}, \{a, b, c\}, \{a, b, d\}, X$

superset of $\{a, b, d\}$ are: $\{a, b, d\}, X$

$\boldsymbol{N_a}$ =Collection of T nbds of point a

= Collection of all distinct supersets of T-open sets containing point a

$= \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$

To find neighbourhood system of $\boldsymbol{b}$:

T-open sets containing point b are : $X, \{a, b\}, \{a, b, d\}$

superset of X is X

superset of $\{a, b\}$ are: $\{a, b\}, \{a, b, c\}, \{a, b, d\}, X$

superset of $\{a, b, d\}$ are: $\{a, b, d\}, X$

$\boldsymbol{N_b}$ =Collection of T nbds of point b

$=$ Collection of all distint supersets of T-open sets containing point b

$=\big\{\{a,b\},\{a,b,c\},\{a,b,d\},X\big\}$

Example 2. If $X=\{1,2,3,4,5\}$

And $T=\{X,\emptyset,\{1\},\{1,2\},\{1,2,5\},\{1,34\},\{1,2,34\},\}$ be a topology then find $N(1)$

To find neighbourhood system of 1:

T-open sets containing point 1 are : $X,\{1\},\{1,2\},\{1,2,5\},\{1,34\},\{1,2,34\}$

And superset of X is X

superset of $\{1\}$ are :
$X,\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,2,3\},\{1,2,4\}.\{1,2,5\},\{1,3,4\},\{1,3,5\}$

$\{1,4,5\},\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\}$

superset of $\{1,2\}$ are:

$X,\{1,2\},\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},$

superset of $\{1,2,5\}$ are: $\{1,2,5\},\{1,2,3,5\},\{1,2,4,5\},X$

superset of $\{1,3,4\}$ are: $\{1,3,4\},\{1,2,3,4\},\{1,3,4,5\},X$

superset of $\{1,2,3,4\}$ are: $\{1,2,3,4\},X$

$N(1)=$ Collection of all T nbds of point 1

$=$ Collection of all distint supersets of T-open sets containing point 1

$=\{X,\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,2,3\},\{1,2,4\}.\{1,2,5\},\{1,3,4\},\{1,3,5\}$

$\{1,4,5\},\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\}\}$

Theorem- A subset of a Topological space is open iff it is neighbourhood of each of its points.

Proof. Let (X,T) be a topological space and G be a subset of X. Then

G is $T - open \Rightarrow \forall x \in G, x \in G \subseteq G$

$\Rightarrow G$ is a $T - nbd$ of $x, \forall x \in G$

$\Rightarrow G$ is a nbd of each of its point.

Conversely let G be a nbd of each of its point,then

If $x \in G$, $\exists$ a T-open set say G_x such that $x \in G_x \subset G$.

$\Rightarrow \bigcup\{ x: x \in G\} \subset \bigcup\{ G_x: x \in G\} \subset \bigcup G$

$\Rightarrow G \subset \bigcup\{ G_x: x \in G\} \subset G$

$\Rightarrow G = \bigcup\{ G_x: x \in G\}$

Thus G is a union of open sets, hence G is open.

Closed Sets in a topological space

Let (X, T) be a topological space, then a subset F of X is said to be closed set or T -closed set if and only if its complement $F' = X - F$ is an open set(i.e. T -open set).

Examples:

1. $\emptyset$ and X are closed sets.

2. If $X = \{1,2,3,4,5\}$

And $T = \{X, \emptyset, \{1\}, \{1,2\}, \{1,2,5\}, \{1,34\}, \{1,2,34\}, \}$ be a topology then T-closed subsets of X are: $X' = \emptyset$

$\emptyset' = X$

$\{1\}' = \{2,3,4,5\}$

$\{1,2\}' = \{3,4,5\}$

$\{1,2,5\}' = \{3,4\}$

$\{1,3,4\}' = \{2,5\}$

$$\{1,2,3,4\}' = \{5\}$$

3. In case of a discrete topological space every subset of X is a closed set.

Door space

Let (X, T) be a topological space then it is said to be a door space if and only if every subset of X is either open or closed.

Example: If $X = \{a, b, c\}$

$$T = \{X, \emptyset, \{b\}, \{a, b\}, \{b, c\}\}$$

$$P(X) = \{ X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\{a, c\}$$

$$X' = \emptyset$$

$$\emptyset' = X$$

$$\{b\}' = \{a, c\}$$

$$\{a, b\}' = \{c\}$$

$$\{b, c\}' = \{a\}$$

Thus closed sets of X are : $\emptyset$, X, $\{a, c\}$,$\{c\}$,$\{a\}$

And open sets of X are: $X, \emptyset, \{b\}, \{a, b\}, \{b, c\}$

Obviously every subset of X is either open or closed.

$\therefore$ X is a door space.

Union and intersection of closed sets

Theorem 1 –If $F_1 and\ F_2$ are two closed subsets of a topological space X then $F_1 \cup F_2$ is a closed set.

Proof – Let $F_1 and\ F_2$ be two closed subsets of a topological space X then

$F_1 and\ F_2$ are closed $\Rightarrow F'_1 and\ F'_2$ are open

$\Rightarrow F'_1 \cap F'_2$ is open [by T_2

$\Rightarrow (F_1 \cup F_2)'$ is open

$\Rightarrow F_1 \cup F_2$ is closed Hence Proved

Theorem 2 –If $\{F_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of closed subsets of a topological space then $\bigcap_{\lambda \in \Lambda} F_\lambda$ is a closed set.

Proved: Let $\{F_\lambda : \lambda \in \Lambda\}$ } be an arbitrary collection of closed subsets of a topological space X then

F_λ is closed $\forall \lambda \in \Lambda \Rightarrow F'_\lambda$ is open $\forall \lambda \in \Lambda$

$\Rightarrow \cup \{F'_\lambda : \lambda \in \Lambda\}$ is open [by T_3

$\Rightarrow [\cap \{F_\lambda : \lambda \in \Lambda\}]'$ is open [by De Morgan's law

$\Rightarrow [\cap \{F_\lambda : \lambda \in \Lambda\}]$ is closed. Hence Proved

Characrterisation of a Topological space in Terms of closed sets

Theorem : Let X be a non-empty set and $\boldsymbol{F}$ a family of subsets of X such that

$[F_1]$: $\emptyset \in \boldsymbol{F}, X \in \boldsymbol{F}$

$[F_2]$: $F_1, F_2 \in \boldsymbol{F} \Rightarrow F_1 \cup F_2 \in \boldsymbol{F}$

$[F_3]$: $F_\lambda \in \boldsymbol{F}, \forall \lambda \in \Lambda \Rightarrow [\cap \{F_\lambda : \lambda \in \Lambda\}] \in \boldsymbol{F}$

Then $\exists$ a unique topology T for X such that T −closed subsets of X are precisely the members of $\boldsymbol{F}$.

Proof –Existence of topology: Let $\boldsymbol{T}$ =collection of complements of sets of $\boldsymbol{F}$

i.e. $\boldsymbol{T} = \{F' : F \in \boldsymbol{F}\}$

then we shall show that T is a topology for X.

$[T_1]$: from $[F_1]$ we have $\emptyset \in \boldsymbol{F}, X \in \boldsymbol{F}$

$\Rightarrow \emptyset' \in \boldsymbol{T}, X' \in \boldsymbol{T}$

$\Rightarrow X \in \boldsymbol{T}, \emptyset \in \boldsymbol{T}$

Thus T_1 is satisfied.

$[T_2]$: Let $G_1, G_2 \in \boldsymbol{T}$

Then $G_1, G_2 \in \boldsymbol{T} \Rightarrow G'_1, G'_2 \in \boldsymbol{F}$

$\Rightarrow G'_1 \cup G'_2 \in \boldsymbol{F}$

$\Rightarrow (G_1 \cap G_2)' \in \boldsymbol{F}$

$\Rightarrow G_1 \cap G_2 \in \boldsymbol{T}$

$\therefore T_2$ is satisfied

$[T_3]$: Let $\{G_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of members of $\boldsymbol{T}$

Then $G_\lambda \in \boldsymbol{T} \ \ \forall \lambda \in \Lambda \Rightarrow G'_\lambda \in \boldsymbol{F} \ \ \forall \lambda \in \Lambda$

$\Rightarrow \cap \{G'_\lambda : \lambda \in \Lambda\} \in \boldsymbol{F}$

$\Rightarrow [\cup \{G_\lambda : \lambda \in \Lambda\}]' \in \boldsymbol{F}$

$\Rightarrow \cup \{G_\lambda : \lambda \in \Lambda\} \in \boldsymbol{T}$

$\therefore T_3$ is satisfied

Hence $\boldsymbol{T}$ is a topology on X.

$\therefore$ Any subset $F \subset X$ is $\boldsymbol{T}$ −closed iff $F' \in \boldsymbol{F}$

Uniqueness of topology: Suppose that $\boldsymbol{T}$ and $\boldsymbol{T}'$ be two topologies having the same system of closed sets. Then

$G \in T \Leftrightarrow G$ is $\boldsymbol{T} -$ open

$\Leftrightarrow G'$ is $\boldsymbol{T} -$ closed

$\Leftrightarrow G'$ is $\boldsymbol{T}' -$ closed [since $\boldsymbol{T}$ and $\boldsymbol{T}'$ have the same system of closed sets.]

$\Leftrightarrow G$ is $\boldsymbol{T'}-$ open

Hence $\boldsymbol{T} = \boldsymbol{T'}$

Characrterisation of a Topological space in Terms of Neighbourhood

Theorem : Let X be a non-empty set and with each $x \in X$ let there be associated a family $\boldsymbol{N(x)}$ of subsets of X called neighbourhoods satisfying the following conditions:

$[N_0]$:$\boldsymbol{N(x)} \neq \emptyset \;\; \forall x \in X$

$[N_1]$: $N \in \boldsymbol{N(x)} \Rightarrow x \in N$

$[N_2]$: $N \in \boldsymbol{N(x)}, M \supset N \Rightarrow M \in \boldsymbol{N(x)}$

$[N_3]$: $N \in \boldsymbol{N(x)}, M \in \boldsymbol{N(x)} \Rightarrow N \cap M \in \boldsymbol{N(x)}$

$[N_4]$: $N \in \boldsymbol{N(x)} \Rightarrow \exists M \in \boldsymbol{N(x)}$ $s.t. M \subset N$ and $M \in \boldsymbol{N(y)}$ $\forall y \in M$

Then $\exists$ a unique topology $\boldsymbol{T}$ for X in such a way that if $\boldsymbol{N^*(x)}$ **is the collection of nbd of** x, defined by the topology $\boldsymbol{T}$ then $\boldsymbol{N(x)} = \boldsymbol{N^*(x)}$

Proof – Let $\boldsymbol{T}$ =collection of complements of sets of $\boldsymbol{F}$

i.e. $\boldsymbol{T} = \{G \in N(x) : \forall x \in G\}$

then we shall show that T is a topology on X.

$[T_1]$: $\emptyset \in \boldsymbol{T}$ [since $\emptyset$ contains no points so the statement $\emptyset \in N(x)$ $\forall x \in \emptyset$ is trivially true]

We now show that $X \in T$

From $[N_0]$ $\boldsymbol{N(x)} \neq \emptyset \;\; \forall x \in X$

$\therefore$ $\exists$ some set $G_x \in N(x)$ $\forall x \in X$

And $X \supset G_x$ $\forall x \in X$

$\therefore$ by $[N_2]$ it follows that $X \in \boldsymbol{N(x)}$ $\forall x \in X$

Hence $X \in \boldsymbol{T}$

Thus T_1 is satisfied.

$[T_2]$: Let $G_1, G_2 \in \boldsymbol{T}$

Then $G_1, G_2 \in \boldsymbol{T} \Rightarrow G_1 \in \boldsymbol{N}(\boldsymbol{x})\ \forall x \in G_1\ and\ G_2 \in \boldsymbol{N}(\boldsymbol{x})\ \forall x \in G_2$

$\Rightarrow G_1 \in \boldsymbol{N}(\boldsymbol{x}), G_2 \in \boldsymbol{N}(\boldsymbol{x})\ \forall x \in G_1 \cap G_2$

$\Rightarrow (G_1 \cap G_2) \in \boldsymbol{N}(\boldsymbol{x})\ \ \forall x \in G_1 \cap G_2$ [by $[N_3]$

$\Rightarrow G_1 \cap G_2 \in \boldsymbol{T}$ **[**by def. of $\boldsymbol{T}$

$\therefore T_2$ is satisfied

$[T_3]$: Let $\{G_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of members of $\boldsymbol{T}$ and let x be any arbitrary element of $\cup \{G_\lambda : \lambda \in \Lambda\}$

Then $x \in G_\lambda\ \ for\ \ some\ \lambda \in \Lambda$ and $G_\lambda \in \boldsymbol{T} \Rightarrow G_\lambda \in \boldsymbol{N}(\boldsymbol{x})$

Now $G_\lambda \in \boldsymbol{N}(\boldsymbol{x})$ and $\cup \{G_\lambda : \lambda \in \Lambda\} \supset G_\lambda$

$\Rightarrow \cup \{G_\lambda : \lambda \in \Lambda\} \in \boldsymbol{N}(\boldsymbol{x})\ \ \forall x \in \cup \{G_\lambda : \lambda \in \Lambda\}$ **[**by $[N_2]$

$\Rightarrow \cup \{G_\lambda : \lambda \in \Lambda\} \in \boldsymbol{T}$ **[** by def. of $\boldsymbol{T}$

$\therefore T_3$ is satisfied

Hence $\boldsymbol{T}$ is a topology on X.

$To\ prove\ that\ \boldsymbol{N}(\boldsymbol{x}) = \boldsymbol{N}^*(\boldsymbol{x})$ **:** Let $N \in \boldsymbol{N}(\boldsymbol{x})$ by $[N_4]$ there exists $M \in \boldsymbol{N}(\boldsymbol{x})$ **such that**

$M \subset N$ and $M \in \boldsymbol{N}(\boldsymbol{y})\ \ \forall y \in M$

Now $M \in \boldsymbol{N}(\boldsymbol{x}) \Rightarrow x \in M$

And $M \in \boldsymbol{N}(\boldsymbol{y})\ \ \forall y \in M \Rightarrow M \in \boldsymbol{T}$

Thus M is a $\boldsymbol{T}$ $-$open set such that $x \in M \subset N$. It follows that N is a $\boldsymbol{T}$nbd of x i.e. $N \in \boldsymbol{N}^*(\boldsymbol{x})$

Hence $\boldsymbol{N}(\boldsymbol{x}) \subset \boldsymbol{N}^*(\boldsymbol{x})$ ……………………….(i)

Conversely let $N \in \boldsymbol{N}^*(\boldsymbol{x})$ so that N is a $\boldsymbol{T}$ nbd of x. Then there exists a T open set G such that $x \in G \subset N$

Now $G \in T \Rightarrow G \in \boldsymbol{N}(\boldsymbol{x}) \ \forall x \in G$

But $G \in \boldsymbol{N}(\boldsymbol{x})$ **and** $G \subset N \Rightarrow N \in \boldsymbol{N}(\boldsymbol{x})$ **[by** $[N_2]$

Hence $\boldsymbol{N}^*(x) \subset \boldsymbol{N}(x)$ **……………………….(ii)**

From (i) and (ii) we get $\boldsymbol{N}(\boldsymbol{x}) = \boldsymbol{N}^*(\boldsymbol{x})$ Hence proved