Countable and uncountable sets

Theorem –A set A is countably infinite (denumerable) iff A can be put in the form of a sequence $\{a_1, a_2, a_3, \ldots\}$ of distinct elements.

Proof- Suppose A is countably infinite, then

$$A \text{ is countably infinite } \Rightarrow \exists \text{ a one one onto map } f: N \to A$$

$$\Rightarrow \text{for } m \neq n, f(m) \neq f(n) \qquad [\text{where } n, m \in N]$$

If we define $f: N \to A$ by

$$f(n) = a_n \quad \forall n \in N$$

Then $m \neq n \Rightarrow f(m) \neq f(n)$

$$\Rightarrow a_m \neq a_n$$

$\therefore$ $a_1, a_2, a_3, \ldots\ldots a_n, \ldots\ldots$ all are distinct.

Thus elements of set A can be put in the terms of a sequence $\{a_1, a_2, a_3, \ldots\ldots a_n, \ldots.\}$ of distinct elements.

Theorem (Galileo paradox) –Any denumerable set can be put into a one one correspondence with a proper subset of itself.

Proof – Let a set A be denumerable. Then all the elements of A can be written as a sequence

$$a_1, a_2, a_3, \ldots\ldots a_n, \ldots\ldots$$

Let $B = A - \{a_1\}$

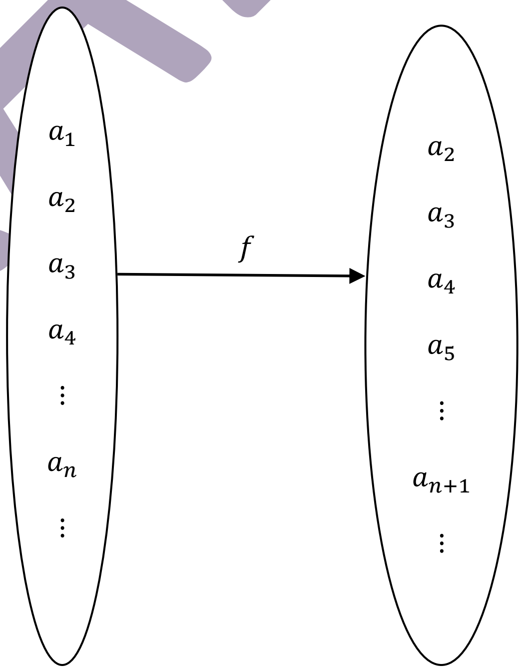
Evidently B is a proper subset of set A.

Consider the mapping

$$f: A \to B$$

Given by $f(a_i) = a_{i+1} \quad i = 1,2,3, \ldots$

Obviously f is one one and onto.



Hence a denumerable set can be put into a one one correspondence with a proper subset of itself.

Theorem –Every subset of a countable set is countable.

Proof – Let a set A be countable, then set A is either finite or countably infinite (denumerable).

(i) When A is finite

A is finite $\Rightarrow$ Every subset$B$ of A is finite

$\Rightarrow$ Every subset of A is countable.

(ii)) When A is countably infinite (denumerable)

A can be written as a sequence of distinct terms, say $A = \{a_1, a_2, a_3, \ldots\ldots a_n, \ldots.\}$.

Then two cases arise for $B \subset A$

Either $B = \emptyset$ Or $B \neq \emptyset$

If $B = \emptyset$,It is countable.

If $B \neq \emptyset$, let n_1 be the smallest positive integer such that $a_{n_1} \in B$.

And let n_2 be the smallest positive integer greater than n_1 , such that $a_{n_2} \in B$.

In the same way ,positive integers $n_3, n_4, \ldots n_k, \ldots$ can be chosen in succession such that $a_{n_k} \in B, k = 3,4,5, \ldots$

So we can define a mapping

$$f: N \to B$$

Given by $$f(k) = a_{n_k} \quad k = 1,2.3 \ldots.$$

Which is one one onto, therefore $B \sim N$.Hence B is countably infinite i.e. countable.

Theorem – The Union of a countable family of denumerable set is denumerable.

Proof – Let $\{A_n : n \in N\}$ be a countable family of denumerable sets. Then

A_n is denumerable $\Rightarrow$ Elements of set A_n can be put in a sequence, therefore we have

$$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\ldots a_{nn}, \ldots.\}. \quad \forall n \in N$$

Now make a list of all the elements of sets $A_n, n = 1,2,3,4\ldots\ldots\ldots$as follows:

$$\begin{array}{lllllll}
A_1 & a_{11} & a_{12} & a_{13} & a_{14} & \ldots & \ldots \\
A_2 & a_{21} & a_{22} & a_{23} & a_{24} & \ldots & \ldots \\
A_3 & a_{31} & a_{32} & a_{33} & a_{34} & \ldots & \ldots \\
A_4 & a_{41} & a_{42} & a_{43} & a_{44} & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
A_n & a_{n1} & a_{n2} & a_{n3} & a_{n4} & \ldots & \ldots
\end{array}$$

Then starting from the upper left corner of this array and taking the elements diagonally, we can list all the elements of $\cup A_n$ as follows:

a_{11}

a_{21} a_{12}

a_{31} a_{22} a_{13}

a_{41} a_{32} a_{23} a_{14}

… … … … …

… … … … … …

a_{n1} $a_{(n-1)2}$ $a_{(n-2)3}$ … … … a_{1n}

… … … … … … … …

… … … … … … … … …

Evidently in this list, we can observe that $a_{pq} = q^{th}\ element\ of\ (p+q-1)^{th}$ row. Thus all the elements of $\cup A_n$, have been arranged in an infinite sequence as

$$a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, a_{41}, a_{32}, a_{23}, a_{14} \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence $\cup A_n$ is countable.

Note – A map $f\colon \cup A_n \to N$ defined by $f\left(a_{pq}\right) = \frac{(p+q-2)(P+q-1)}{2} + q$ is a bijective map between $\cup A_n$ and N.

Problem –(i) Prove that the unit interval $[0,1]$ is uncountable.

(ii) Prove that the set of all real numbers R is uncountable.

Solution : (i) We have to prove that the unit interval $[0,1]$ is uncountable

Assume the contrary , i.e. the unit interval $[0,1]$ is countable, then elements of set $[0,1]$ can be written in a sequence as

$$[0,1] = \{x_1, x_2, x_3, \ldots., \ldots\}$$

So that every number in $[0,1]$ occurs among the x_i , $i \in N$.

Now we can express each number in $[0,1]$ in the the form of an infinite decimal as follows:

$$x_1 = 0.a_{11}a_{12}a_{13}a_{14} \ldots\ldots\ldots a_{1n} \ldots\ldots$$

$$x_2 = 0.a_{21}a_{22}a_{23}a_{24} \ldots\ldots\ldots a_{2n} \ldots\ldots$$

$$x_3 = 0.a_{31}a_{32}a_{33}a_{34} \ldots\ldots\ldots a_{3n} \ldots\ldots$$

$$x_4 = 0.a_{41}a_{42}a_{43}a_{44} \ldots\ldots\ldots a_{4n} \ldots\ldots$$

$$x_n = 0.a_{n1}a_{n2}a_{n3}a_{n4} \ldots\ldots\ldots a_{nn} \ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Where $a_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}$

Now let us construct a real number

$$x = 0.b_1b_2b_3b_4 \ldots\ldots\ldots b_n \ldots$$

Such that $b_1 \neq a_{11}, b_2 \neq a_{22}, b_3 \neq a_{33}, b_4 \neq a_{44}, \dots\dots\dots b_n \neq a_{nn}$ ...etc.

Obviously $x \neq x_1, x \neq x_2, x \neq x_3 \dots\dots x \neq x_n$ for $n \in N$ and $\in [0,1]$. This shows that all the elements of $[0,1]$ cannot be listed in a sequence ,so our assumption is wrong. Hence the unit interval $[0,1]$ is uncountable.

(ii) We have to prove that the set of all real numbers R is uncountable.

Suppose that R is countable, since $[0,1]$ is an infinite subset of R.

R is countable $\Rightarrow [0,1]$ is countable. [$\because$ Every subset of a countable set is countable]

But above result of (i) $\Rightarrow [0,1]$ is uncountable

So our assumption is wrong. Hence the set of all real number is uncountable.

Theorem – If sets A and B are countable then $A \times B$ is also countable.

Proof - $\because$ Sets A and B are countable , elements of A and B can be listed as follows:

$$A = \{a_1, a_2, a_3, \dots\dots a_n, \dots.\}.$$

$$B = \{b_1, b_2, b_3, \dots\dots b_n, \dots.\}.$$

Then the elements of $A \times B$ can be listed as follows:

$(a_1, b_1),$

$(a_2, b_1), (a_1, b_2),$

$(a_3, b_1), (a_2, b_2), (a_1, b_3),$

$(a_4, b_1), (a_3, b_2), (a_2, b_3), (a_1, b_4)$

$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$

$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$

$(a_n, b_1), (a_{n-1}, b_2), (a_{n-2}, b_3), \dots \dots (a_2, b_{n-1}), (a_1, b_n)$

$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$

It can be observed that in the list

$$(a_p, b_q) = q^{th}\ element\ of\ (p+q-1)^{th} \text{ row.}$$

Thus all the elements of $A \times B$ have been arranged in an infinite sequence as

$A \times B = \{(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_3, b_1), (a_2, b_2), (a_1, b_3), \dots\}$

Hence $A \times B$ is countable.

Note – A map $f: A \times B \to N$ defined by $f(a_p, b_q) = \frac{(p+q-2)(P+q-1)}{2} + q$ is a bijective mapping between $A \times B$ and N.